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Functional determinants for radial operators

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Abstract

We derive simple new expressions, in various dimensions, for the functional determinant of a radially separable partial differential operator, thereby generalizing the one-dimensional result of Gel'fand and Yaglom to higher dimensions. We use the zeta function formalism, and the results agree with what one would obtain using the angular momentum cutoff method based on radial WKB. The final expression is numerically equal to an alternative expression derived in a Feynman diagrammatic approach, but is considerably simpler.

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1. Introduction and results

Determinants of differential operators occur naturally in many applications in mathematical and theoretical physics, and also have inherent mathematical interest since they encode certain spectral properties of differential operators. Physically, such determinants arise in semiclassical and one-loop approximations in quantum mechanics and quantum field theory [1–3]. Determinants of free Laplacians and free Dirac operators have been extensively studied [4–10], but much less is known about operators involving an arbitrary potential function. For ordinary (i.e., one dimensional) differential operators, a general theory has been developed for determinants of such operators [10–15]. In this paper we extend these results to a broad class of separable *partial* differential operators. The result for four dimensions was found previously in [16] using radial WKB and an angular momentum cut-off regularization and renormalization [17]. Here we re-derive this result using the zeta function approach to determinants [4, 5, 7, 10], and generalize to other dimensions. We are motivated by applications in quantum field theory, so we concentrate on the dimensions $d = 2, 3, 4$, but the mathematical extension to higher dimensions is immediate. We also compare with another expression for the four-dimensional determinant [18], derived using a Feynman diagrammatic approach.

Consider the radially separable partial differential operators

$$\mathcal{M} = -\Delta + V(r) \quad (1.1)$$

$$\mathcal{M}^{\text{free}} = -\Delta, \quad (1.2)$$

where Δ is the Laplace operator in \mathbb{R}^d , and $V(r)$ is a radial potential vanishing at infinity as $r^{-2-\epsilon}$ for $d = 2$ and $d = 3$, and as $r^{-4-\epsilon}$ for $d = 4$. For $d = 1$, with Dirichlet boundary conditions on the interval $[0, \infty)$, the results of Gel'fand and Yaglom [11] lead to the following simple expression for the determinant ratio:

$$\frac{\det[\mathcal{M} + m^2]}{\det[\mathcal{M}^{\text{free}} + m^2]} = \frac{\psi(\infty)}{\psi^{\text{free}}(\infty)}, \quad (1.3)$$

where $[\mathcal{M} + m^2]\psi = 0$, with *initial value* boundary conditions: $\psi(0) = 0$ and $\psi'(0) = 1$. The function ψ^{free} is defined similarly in terms of the free operator: $[\mathcal{M}^{\text{free}} + m^2]$. The squared mass, m^2 , is important for physical applications, and plays the mathematical role of a spectral parameter. The result (1.3) is geometrically interesting, in addition to being computationally simple, as it means that the determinant is determined simply by the *boundary values* of the solutions of $[\mathcal{M} + m^2]\psi = 0$, and no detailed information is needed concerning the actual spectrum of eigenvalues.

Now consider dimensions greater than 1 (as mentioned, we are most interested in $d = 2, 3, 4$; but the extension to higher dimensions is immediate). Since the potential is radial, $V = V(r)$, we can express the eigenfunctions of \mathcal{M} as linear combinations of basis functions of the form

$$\Psi(r, \vec{\theta}) = \frac{1}{r^{(d-1)/2}} \psi_{(l)}(r) Y_{(l)}(\vec{\theta}), \quad (1.4)$$

where $Y_{(l)}(\vec{\theta})$ is a hyperspherical harmonic [19], labelled in part by a non-negative integer l , and the radial function $\psi_{(l)}(r)$ is an eigenfunction of the Schrödinger-like radial operator

$$\mathcal{M}_{(l)} \equiv -\frac{d^2}{dr^2} + \frac{(l + \frac{d-3}{2})(l + \frac{d-1}{2})}{r^2} + V(r). \quad (1.5)$$

$\mathcal{M}_{(l)}^{\text{free}}$ is defined similarly, with the potential omitted: $V = 0$. In dimension $d \geq 2$, the radial eigenfunctions $\psi_{(l)}$ have degeneracy given by [19]

$$\text{deg}(l; d) \equiv \frac{(2l + d - 2)(l + d - 3)!}{l!(d - 2)!}. \quad (1.6)$$

Formally, for the separable operators in (1.1)–(1.2), the logarithm of the determinant ratio can be written as a sum over l (weighted with the degeneracy factor) of the logarithm of one-dimensional determinant ratios,

$$\ln \left(\frac{\det[\mathcal{M} + m^2]}{\det[\mathcal{M}^{\text{free}} + m^2]} \right) = \sum_{l=0}^{\infty} \text{deg}(l; d) \ln \left(\frac{\det[\mathcal{M}_{(l)} + m^2]}{\det[\mathcal{M}_{(l)}^{\text{free}} + m^2]} \right). \quad (1.7)$$

Each term in the sum can be computed using the Gel'fand–Yaglom result (1.3). However, the l sum in (1.7) is divergent, as noted by Forman [13] for the free Laplace operator in a two-dimensional disc. In this paper we show how to define a finite and renormalized determinant ratio for the radially separable partial differential operators (1.1)–(1.2). Specifically, we derive the following simple expressions, which generalize (1.3) to higher dimensions:

$$\ln \left(\frac{\det[\mathcal{M} + m^2]}{\det[\mathcal{M}^{\text{free}} + m^2]} \right) \Bigg|_{d=2} = \ln \left(\frac{\psi_{(0)}(\infty)}{\psi_{(0)}^{\text{free}}(\infty)} \right) + \sum_{l=1}^{\infty} 2 \left\{ \ln \left(\frac{\psi_{(l)}(\infty)}{\psi_{(l)}^{\text{free}}(\infty)} \right) - \frac{\int_0^{\infty} dr r V(r)}{2l} \right\} + \int_0^{\infty} dr r V \left[\ln \left(\frac{\mu r}{2} \right) + \gamma \right] \quad (1.8)$$

$$\ln \left(\frac{\det[\mathcal{M} + m^2]}{\det[\mathcal{M}^{\text{free}} + m^2]} \right) \Big|_{d=3} = \sum_{l=0}^{\infty} (2l+1) \left\{ \ln \left(\frac{\psi_{(l)}(\infty)}{\psi_{(l)}^{\text{free}}(\infty)} \right) - \frac{\int_0^{\infty} dr r V(r)}{2(l + \frac{1}{2})} \right\} \quad (1.9)$$

$$\begin{aligned} \ln \left(\frac{\det[\mathcal{M} + m^2]}{\det[\mathcal{M}^{\text{free}} + m^2]} \right) \Big|_{d=4} &= \sum_{l=0}^{\infty} (l+1)^2 \left\{ \ln \left(\frac{\psi_{(l)}(\infty)}{\psi_{(l)}^{\text{free}}(\infty)} \right) - \frac{\int_0^{\infty} dr r V(r)}{2(l+1)} \right. \\ &\quad \left. + \frac{\int_0^{\infty} dr r^3 V(V+2m^2)}{8(l+1)^3} \right\} - \frac{1}{8} \int_0^{\infty} dr r^3 V(V+2m^2) \left[\ln \left(\frac{\mu r}{2} \right) + \gamma + 1 \right]. \end{aligned} \quad (1.10)$$

Here γ is Euler's constant, and μ is a renormalization scale (defined in the next section), which is essential for physical applications, and which arises naturally in even dimensions. A conventional renormalization choice is to take $\mu = m$ in (1.8)–(1.10). In each of (1.8)–(1.10), the sum over l is convergent once the indicated subtractions are made. The function $\psi_{(l)}(r)$ is the solution to the radial equation

$$[\mathcal{M}_{(l)} + m^2]\psi_{(l)}(r) = 0, \quad \psi_{(l)}(r) \sim r^{l+(d-1)/2}, \quad \text{as } r \rightarrow 0. \quad (1.11)$$

The function $\psi_{(l)}^{\text{free}}(r)$ is defined similarly, with the same behaviour as $r \rightarrow 0$, in terms of the operator $[\mathcal{M}_{(l)}^{\text{free}} + m^2]$. Thus, in d dimensions, $\psi_{(l)}^{\text{free}}(r)$ is expressed as a Bessel function:

$$\psi_{(l)}^{\text{free}}(r) = \left(\frac{2}{m} \right)^{l+d/2-1} \Gamma \left(l + \frac{d}{2} \right) r^{1/2} I_{l+d/2-1}(mr). \quad (1.12)$$

Note that the results (1.8)–(1.10) state once again that the determinant is determined by the boundary values of solutions of $[\mathcal{M} + m^2]\psi = 0$, with the only additional information being a finite number of integrals involving the potential $V(r)$. We also stress the computational simplicity of (1.8)–(1.10), as the initial value problem (1.11) is trivial to implement numerically. The $d = 4$ result (1.10) was found previously in [16] using radial WKB and an angular momentum cutoff regularization and renormalization [17]. Here we present a different proof using the zeta function approach, and generalize to other dimensions. In fact, the $d = 2$ and $d = 3$ results can also be derived using the radial WKB method of [16, 17]. Furthermore, in section 3 we show how these results also agree with the Feynman diagrammatic approach, by showing that the $d = 4$ zeta function expression (1.10) agrees precisely with a superficially different, and more complicated, $d = 4$ expression found in [18]. Finally, we note that the results in (1.8)–(1.10) are for a *generic* radial potential $V(r)$. There are important physical applications where the potential $V(r)$ is such that the operator \mathcal{M} has negative and/or zero modes [16], in which case these expressions are modified slightly, as in [16] and as discussed below in the conclusions.

2. Zeta function formalism

The functional determinant can be defined in terms of a zeta function [4, 5, 7] for the operator \mathcal{M} . For dimensional reasons, we define

$$\zeta_{[\mathcal{M}+m^2]/\mu^2}(s) = \mu^{2s} \zeta_{[\mathcal{M}+m^2]}(s) = \mu^{2s} \sum_{\lambda} (\lambda + m^2)^{-s}, \quad (2.1)$$

where the sum is over the spectrum of \mathcal{M} , and μ is an arbitrary parameter with dimension of a mass. Physically, μ plays the role of a renormalization scale. Then the logarithm of the determinant is defined as [4, 5, 7]

$$\begin{aligned} \ln \det[\mathcal{M} + m^2] &\equiv -\zeta'_{[\mathcal{M}+m^2]/\mu^2}(0) \\ &= -\ln(\mu^2) \zeta_{[\mathcal{M}+m^2]}(0) - \zeta'_{[\mathcal{M}+m^2]}(0). \end{aligned} \quad (2.2)$$

To compute the determinant *ratio*, we define the zeta function difference

$$\zeta(s) \equiv \zeta_{[\mathcal{M}+m^2]}(s) - \zeta_{[\mathcal{M}^{\text{free}}+m^2]}(s). \quad (2.3)$$

Thus we need to compute the zeta function and its derivative, each evaluated at $s = 0$. In general, the zeta function at $s = 0$ is related to the heat kernel coefficient, $a_{d/2}(\mathcal{P})$, associated with the operator \mathcal{P} [10, 20]:

$$\zeta_{\mathcal{P}}(0) = a_{d/2}(\mathcal{P}). \quad (2.4)$$

For the operator $\mathcal{P} = -\Delta - E$, these heat kernel coefficients are [10] given in \mathbb{R}^d by

$$a_1(\mathcal{P}) = \frac{1}{(4\pi)^{d/2}} \int_{\mathbb{R}^d} d^d x E \quad (2.5)$$

$$a_{3/2}(\mathcal{P}) = 0 \quad (2.6)$$

$$a_2(\mathcal{P}) = \frac{1}{(4\pi)^{d/2}} \int_{\mathbb{R}^d} d^d x \frac{E^2}{2}. \quad (2.7)$$

Thus, setting $E = -V(r) - m^2$, and $E^{\text{free}} = -m^2$, we find

$$\zeta(0) = \begin{cases} -\frac{1}{2} \int_0^\infty dr r V(r), & d = 2 \\ 0, & d = 3 \\ \frac{1}{16} \int_0^\infty dr r^3 V(V + 2m^2), & d = 4. \end{cases} \quad (2.8)$$

The derivative of the zeta function at $s = 0$, $\zeta'(0)$, can be evaluated using the relation to the Jost functions of scattering theory [21, 22]; for the application of these ideas to the Casimir effect see [23]. Consider the radial eigenvalue equation

$$\mathcal{M}_{(l)}\phi_{(l),p} = p^2\phi_{(l),p}, \quad (2.9)$$

where $\mathcal{M}_{(l)}$ is the Schrödinger-like radial operator defined in (1.5). A distinguished role is played by the so-called *regular solution*, $\phi_{(l),p}(r)$, which is defined to have the same behaviour as $r \rightarrow 0$ as the solution *without* potential:

$$\phi_{(l),p}(r) \underset{r \rightarrow 0}{\sim} \hat{j}_{l+(d-3)/2}(pr). \quad (2.10)$$

Here the spherical Bessel function $\hat{j}_{l+(d-3)/2}$ is defined as

$$\hat{j}_{l+(d-3)/2}(z) = \sqrt{\frac{\pi z}{2}} J_{l+d/2-1}(z).$$

The asymptotic behaviour of the regular solution, $\phi_{(l),p}(r)$, as $r \rightarrow \infty$ defines the *Jost function*, $f_l(p)$, [21]

$$\phi_{(l),p}(r) \underset{r \rightarrow 0}{\sim} \frac{i}{2} [f_l(p)\hat{h}_{l+(d-3)/2}^-(pr) - f_l^*(p)\hat{h}_{l+(d-3)/2}^+(pr)]. \quad (2.11)$$

Here $\hat{h}_{l+(d-3)/2}^-(pr)$ and $\hat{h}_{l+(d-3)/2}^+(pr)$ are the Riccati–Hankel functions

$$\hat{h}_{l+(d-3)/2}^+(z) = i\sqrt{\frac{\pi z}{2}} H_{l+d/2-1}^{(1)}(z), \quad \hat{h}_{l+(d-3)/2}^-(z) = -i\sqrt{\frac{\pi z}{2}} H_{l+d/2-1}^{(2)}(z).$$

As is well known from scattering theory [21, 22], the analytic properties of the Jost function $f_l(p)$ strongly depend on the properties of the potential $V(r)$. Analyticity of the Jost function as a function of p for $\text{Im} p > 0$ is guaranteed, if in addition to the aforementioned

behaviour as $r \rightarrow \infty$, we impose $V(r) \sim r^{-2+\epsilon}$ for $r \rightarrow 0$, and continuity of $V(r)$ in $0 < r < \infty$ (except perhaps at a finite number of finite discontinuities). For us, the analytic properties of the Jost function in the upper half plane will be of particular importance because they are related to the shifting of contours in the complex momentum plane.

By standard contour manipulations [10], the zeta function can be expressed in terms of the Jost functions as:

$$\zeta(s) = \frac{\sin \pi s}{\pi} \sum_{l=0}^{\infty} \text{deg}(l; d) \int_m^{\infty} dk [k^2 - m^2]^{-s} \frac{\partial}{\partial k} \ln f_l(ik). \tag{2.12}$$

This representation is valid for $\text{Re } s > d/2$, and the technical problem is the construction of the analytic continuation of (2.12) to a neighbourhood about $s = 0$. If expression (2.12) were analytic at $s = 0$, then we would deduce that

$$\zeta'(0) = - \sum_{l=0}^{\infty} \text{deg}(l; d) \ln f_l(im). \tag{2.13}$$

From the definition (2.11) of the Jost function,

$$f_l(im) = \frac{\phi_{(l),im}(\infty)}{\phi_{(l),im}^{\text{free}}(\infty)} = \frac{\psi_{(l)}(\infty)}{\psi_{(l)}^{\text{free}}(\infty)}, \tag{2.14}$$

where $\psi_{(l)}(r)$ is defined in (1.11). Thus, the regulated expression (2.13) coincides with the formal partial wave expansion (1.7), using the Gel'fand–Yaglom result (1.3) for each l . However, the expansion (2.13) is divergent in positive integer dimensions. In the zeta function approach, the divergence of the formal sum in (2.13) is directly related to the need for analytic continuation of $\zeta(s)$ in s to a region including $s = 0$. From (2.12), this analytic continuation relies on the uniform asymptotic behaviour of the Jost function $f_l(ik)$. Denoting this behaviour by $f_l^{\text{asym}}(ik)$, the analytic continuation is achieved by adding and subtracting the leading asymptotic terms of the integrand in (2.12) to write

$$\zeta(s) = \zeta_f(s) + \zeta_{as}(s), \tag{2.15}$$

where

$$\zeta_f(s) = \frac{\sin(\pi s)}{\pi} \sum_{l=0}^{\infty} \text{deg}(l; d) \int_m^{\infty} dk [k^2 - m^2]^{-s} \frac{\partial}{\partial k} [\ln f_l(ik) - \ln f_l^{\text{asym}}(ik)], \tag{2.16}$$

and

$$\zeta_{as}(s) = \frac{\sin(\pi s)}{\pi} \sum_{l=0}^{\infty} \text{deg}(l; d) \int_m^{\infty} dk [k^2 - m^2]^{-s} \frac{\partial}{\partial k} \ln f_l^{\text{asym}}(ik). \tag{2.17}$$

Ultimately we are interested in the analytic continuation of $\zeta(s)$ to $s = 0$. As many asymptotic terms will be included in $f_l^{\text{asym}}(ik)$ as are necessary to make $\zeta_f(s)$ as given in (2.16) analytic around $s = 0$. On the other hand, for $\zeta_{as}(s)$ the analytic continuation to $s = 0$ can be constructed in closed form using an explicit representation of the asymptotic behaviour of the Jost function, derived in the next section.

2.1. Asymptotics of the Jost function

The asymptotics of the Jost function $f_l(ik)$ follows from standard results in scattering theory [21]. The starting point is the integral equation for the regular solution

$$\phi_{(l),p}(r) = \hat{J}_{l+(d-3)/2}(pr) + \int_0^r dr' \mathcal{G}_{l,p}(r, r') V(r') \phi_{(l),p}(r'), \tag{2.18}$$

with Green's function

$$\mathcal{G}_{l,p}(r, r') = \frac{i}{2p} [\hat{h}_{l+(d-3)/2}^-(pr) \hat{h}_{l+(d-3)/2}^+(pr') - \hat{h}_{l+(d-3)/2}^+(pr) \hat{h}_{l+(d-3)/2}^-(pr')]. \quad (2.19)$$

Asymptotically for $r \rightarrow \infty$,

$$\phi_{(l),p}(r) \sim \hat{J}_{l+(d-3)/2}(pr) + \int_0^\infty dr' \mathcal{G}_{l,p}(r, r') V(r') \phi_{(l),p}(r'). \quad (2.20)$$

Noting that $[\hat{h}_\nu^\pm(x)]^* = \hat{h}_\nu^\mp(x)$, for x real, this asymptotic behaviour can be written as

$$\begin{aligned} \phi_{(l),p}(r) \sim \frac{i}{2} \left\{ \left[1 + \frac{1}{p} \int_0^\infty dr' \hat{h}_{l+(d-3)/2}^+(pr') V(r') \phi_{(l),p}(r') \right] \hat{h}_{l+(d-3)/2}^-(pr) \right. \\ \left. - \left[1 + \frac{1}{p} \int_0^\infty dr' \hat{h}_{l+(d-3)/2}^+(pr') V(r') \phi_{(l),p}(r') \right]^* \hat{h}_{l+(d-3)/2}^+(pr) \right\}. \end{aligned}$$

Comparing with the definition (2.11) of the Jost function, we find the following integral equation for the Jost function:

$$f_l(p) = 1 + \frac{1}{p} \int_0^\infty dr \hat{h}_{l+(d-3)/2}^+(pr) V(r) \phi_{(l),p}(r). \quad (2.21)$$

For the zeta function (2.12) we need the Jost function for imaginary argument, so we rotate using the Bessel function properties [24]

$$I_\nu(z) = e^{-\frac{\pi}{2} \nu i} J_\nu(iz), \quad K_\nu(z) = \frac{\pi i}{2} e^{\frac{\pi}{2} \nu i} H_\nu^{(1)}(iz).$$

Thus (2.21) becomes

$$f_l(ik) = 1 + \int_0^\infty dr r V(r) \phi_{(l),ik}(r) K_{l+d/2-1}(kr). \quad (2.22)$$

We define a convenient short hand for the Bessel function index,

$$\nu \equiv l + \frac{d}{2} - 1, \quad (2.23)$$

and write the partial-wave Lippmann-Schwinger integral equation for the regular solution as

$$\phi_{(l),ik}(r) = I_\nu(kr) + \int_0^r dr' r' [I_\nu(kr) K_\nu(kr') - I_\nu(kr') K_\nu(kr)] V(r') \phi_{(l),ik}(r'). \quad (2.24)$$

This Lippmann-Schwinger equation leads to an iterative expansion for $f_l(ik)$ in powers of the potential $V(r)$. For dimensions $d \leq 4$, we need at most the $O(V)$ and $O(V^2)$ terms of $\ln f_l(ik)$:

$$\begin{aligned} \ln f_l(ik) = \int_0^\infty dr r V(r) K_\nu(kr) I_\nu(kr) - \int_0^\infty dr r V(r) K_\nu^2(kr) \\ \times \int_0^r dr' r' V(r') I_\nu^2(kr') + \mathcal{O}(V^3). \end{aligned} \quad (2.25)$$

This iterative scheme effectively reduces the calculation of the asymptotics of the Jost function to the known uniform asymptotics of the modified Bessel functions K_ν and I_ν [24]. To the required order, for $\nu \rightarrow \infty$, $k \rightarrow \infty$, with k/ν fixed,

$$\begin{aligned} I_\nu(kr) K_\nu(kr) &\sim \frac{t}{2\nu} + \frac{t^3}{16\nu^3} (1 - 6t^2 + 5t^4) + \mathcal{O}\left(\frac{1}{\nu^4}\right), \\ I_\nu(kr') K_\nu(kr) &\sim \frac{1}{2\nu} \frac{e^{-\nu(\eta(k) - \eta(kr'/r))}}{(1 + (kr/\nu)^2)^{1/4} (1 + (kr'/\nu)^2)^{1/4}} \left[1 + \mathcal{O}\left(\frac{1}{\nu}\right) \right], \end{aligned} \quad (2.26)$$

where $t \equiv 1/\sqrt{1+(kr/v)^2}$, and $\eta(k) \equiv \sqrt{1+(kr/v)^2} + \ln[(kr/v)/(1+\sqrt{1+(kr/v)^2})]$. The r' integration in the term quadratic in V is performed by the saddle point method [10]. We therefore define $\ln f_l^{\text{asym}}(ik)$ as the $O(V)$ and $O(V^2)$ parts of this uniform asymptotic expansion:

$$\begin{aligned} \ln f_l^{\text{asym}}(ik) &\equiv \frac{1}{2v} \int_0^\infty dr \frac{rV(r)}{[1+(kr/v)^2]^{1/2}} + \frac{1}{16v^3} \int_0^\infty dr \frac{rV(r)}{[1+(kr/v)^2]^{3/2}} \\ &\times \left[1 - \frac{6}{[1+(kr/v)^2]} + \frac{5}{[1+(kr/v)^2]^2} \right] - \frac{1}{8v^3} \int_0^\infty dr \frac{r^3V^2(r)}{[1+(kr/v)^2]^{3/2}}. \end{aligned} \quad (2.27)$$

2.2. Computing $\zeta'_f(0)$

By construction, $\zeta_f(s)$, defined in (2.16), is now well defined at $s = 0$, and we find

$$\zeta'_f(0) = - \sum_{l=0}^\infty \text{deg}(l; d) [\ln f_l(im) - \ln f_l^{\text{asym}}(im)]. \quad (2.28)$$

This form is suitable for straightforward numerical computation, as the Jost function $f_l(im)$ can be computed using (1.11) and (2.14), while $\ln f_l^{\text{asym}}(im)$ can be computed using (2.27). With the subtraction of $\ln f_l^{\text{asym}}(im)$ in (2.28), the l sum is now convergent.

However, it is possible to find an even simpler expression. It turns out that the subtraction in (2.28) is an over-subtraction. To see this, expand $\ln f_l^{\text{asym}}(im)$ into its large l behaviour as follows:

$$\begin{aligned} \ln f_l^{\text{asym}}(im) &\sim \frac{1}{2v} \int_0^\infty dr rV(r) - \frac{1}{8v^3} \int_0^\infty dr r^3V(V+2m^2) \\ &+ \frac{1}{2v} \int_0^\infty dr rV(r) \left\{ \left[1 + \left(\frac{mr}{v} \right)^2 \right]^{-1/2} - 1 + \frac{1}{2} \left(\frac{mr}{v} \right)^2 \right\} \\ &+ \frac{1}{16v^3} \int_0^\infty dr \frac{rV(r)}{[1+(mr/v)^2]^{3/2}} \left[1 - \frac{6}{[1+(mr/v)^2]} + \frac{5}{[1+(mr/v)^2]^2} \right] \\ &- \frac{1}{8v^3} \int_0^\infty dr r^3V^2(r) \left\{ \left[1 + \left(\frac{mr}{v} \right)^2 \right]^{-3/2} - 1 \right\}. \end{aligned} \quad (2.29)$$

The first term is $O(\frac{1}{l})$, and the second is $O(\frac{1}{l^3})$, while the remaining terms are all $O(\frac{1}{l^3})$. In dimensions $d \leq 4$, the degeneracy factor $\text{deg}(l; d)$ is at most quadratic in l , and so these last terms are finite when summed over l in (2.28). (In fact, in $d = 2$ and $d = 3$, the $O(\frac{1}{l^3})$ terms are also finite when summed over l .) In the next section we show that these finite terms cancel exactly against corresponding terms arising in the evaluation of $\zeta'_{as}(0)$. Thus, for $\zeta'(0) = \zeta'_f(0) + \zeta'_{as}(0)$, we only actually need to subtract the leading large l terms in (2.29), rather than the full asymptotics in (2.27).

2.3. Computing $\zeta'_{as}(0)$

The explicit form of the asymptotic terms in (2.27) provides the analytic continuation to $s = 0$ of $\zeta_{as}(s)$, as defined in (2.17). The k integrals are done using

$$\int_m^\infty dk [k^2 - m^2]^{-s} \frac{\partial}{\partial k} \left[1 + \left(\frac{kr}{v} \right)^2 \right]^{-\frac{n}{2}} = - \frac{\Gamma(s + \frac{n}{2})\Gamma(1-s)}{\Gamma(n/2)} \frac{(\frac{v}{mr})^n m^{-2s}}{(1 + (\frac{v}{mr})^2)^{s+\frac{n}{2}}}. \quad (2.30)$$

Therefore, we find

$$\begin{aligned} \zeta_{as}(s) = & - \sum_{l=0}^{\infty} \text{deg}(l; d) \left[\int_0^{\infty} dr r^{1+2s} V(r) \left\{ \frac{1}{2} \frac{\Gamma(s + \frac{1}{2})}{\Gamma(s)\Gamma(\frac{1}{2})} \frac{v^{-1-2s}}{(1 + (\frac{mr}{v})^2)^{s+1/2}} \right. \right. \\ & + \frac{1}{16} \frac{\Gamma(s + \frac{3}{2})}{\Gamma(s)\Gamma(\frac{3}{2})} \frac{v^{-3-2s}}{(1 + (\frac{mr}{v})^2)^{s+3/2}} - \frac{3}{8} \frac{\Gamma(s + \frac{5}{2})}{\Gamma(s)\Gamma(\frac{5}{2})} \frac{v^{-3-2s}}{(1 + (\frac{mr}{v})^2)^{s+5/2}} \\ & \left. \left. + \frac{5}{16} \frac{\Gamma(s + \frac{7}{2})}{\Gamma(s)\Gamma(\frac{7}{2})} \frac{v^{-3-2s}}{(1 + (\frac{mr}{v})^2)^{s+7/2}} \right\} \right. \\ & \left. - \int_0^{\infty} dr r^{3+2s} V^2(r) \frac{1}{8} \frac{\Gamma(s + \frac{3}{2})}{\Gamma(s)\Gamma(\frac{3}{2})} \frac{v^{-3-2s}}{(1 + (\frac{mr}{v})^2)^{s+3/2}} \right]. \end{aligned} \tag{2.31}$$

We now subtract sufficiently many terms inside the l sum to ensure the analytic continuation of $\zeta_{as}(s)$ to $s = 0$. The added back terms produce Riemann zeta function terms, such as $\zeta_R(2s + 1)$, whose analytic continuation is immediate. For example, in $d = 4$, where $v = l + 1$, and $\text{deg}(l; 4) = (l + 1)^2 = v^2$, the first term in (2.31) involves the function

$$\mathcal{R}_1(s) \equiv \frac{\Gamma(s + \frac{1}{2})r^{2s}}{\Gamma(s)\Gamma(\frac{1}{2})} \sum_{v=1}^{\infty} \frac{v^{1-2s}}{(1 + (\frac{mr}{v})^2)^{s+1/2}}. \tag{2.32}$$

The analytic continuation of this function to $s = 0$ is

$$\begin{aligned} \mathcal{R}_1(s) = & \frac{\Gamma(s + \frac{1}{2})r^{2s}}{\Gamma(s)\Gamma(\frac{1}{2})} \left[\sum_{v=1}^{\infty} v^{1-2s} \left\{ \left(1 + \left(\frac{mr}{v}\right)^2\right)^{-s-1/2} - 1 + \left(s + \frac{1}{2}\right) \left(\frac{mr}{v}\right)^2 \right\} \right. \\ & \left. + \zeta_R(2s - 1) - \left(s + \frac{1}{2}\right) (mr)^2 \zeta_R(2s + 1) \right]. \end{aligned} \tag{2.33}$$

A straightforward computation yields the derivative at $s = 0$:

$$\mathcal{R}'_1(0) = \sum_{v=1}^{\infty} v \left\{ \left(1 + \left(\frac{mr}{v}\right)^2\right)^{-1/2} - 1 + \frac{1}{2} \left(\frac{mr}{v}\right)^2 \right\} - \frac{1}{2} (mr)^2 \left[\ln\left(\frac{r}{2}\right) + \gamma + 1 \right] + \zeta_R(-1). \tag{2.34}$$

Applying this strategy to the remaining terms in (2.31) leads to

$$\begin{aligned} \zeta'_{as}(0)|_{d=4} = & \frac{1}{8} \int_0^{\infty} dr r^3 V(V + 2m^2) \left[\ln\left(\frac{r}{2}\right) + \gamma + 1 \right] \\ & - \int_0^{\infty} dr r V(r) \left\{ \frac{1}{2} \sum_{v=1}^{\infty} v \left[\left(1 + \left(\frac{mr}{v}\right)^2\right)^{-1/2} - 1 + \frac{1}{2} \left(\frac{mr}{v}\right)^2 \right] + \frac{1}{16} \sum_{v=1}^{\infty} \frac{1}{v} \right. \\ & \left. \times \left[\left(1 + \left(\frac{mr}{v}\right)^2\right)^{-3/2} - 6 \left(1 + \left(\frac{mr}{v}\right)^2\right)^{-5/2} + 5 \left(1 + \left(\frac{mr}{v}\right)^2\right)^{-7/2} \right] \right\} \\ & + \frac{1}{8} \int_0^{\infty} dr r^3 V^2(r) \sum_{v=1}^{\infty} \frac{1}{v} \left[\left(1 + \left(\frac{mr}{v}\right)^2\right)^{-3/2} - 1 \right]. \end{aligned} \tag{2.35}$$

Note that the terms involving summation over v cancel exactly against identical terms in $\zeta'_f(0)$ from (2.29), after those terms are summed over l with the $d = 4$ degeneracy factor $v^2 = (l + 1)^2$. Furthermore, note that the $\ln r$ term inside the integral on the first line of (2.35)

is precisely of the same form as the renormalization term in (2.8), so the $\ln \mu$ in (2.2) combines with $\ln r$ to form the dimensionless combination $\ln(\mu r)$ in (1.10).

The analogous computation in $d = 3$, with degeneracy factor $\text{deg}(l; 3) = (2l + 1) = 2\nu$, leads to

$$\begin{aligned} \zeta'_{as}(0)|_{d=3} = & - \int_0^\infty dr r V(r) \left\{ \sum_{l=0}^\infty \left[\left(1 + \left(\frac{mr}{l + \frac{1}{2}} \right)^2 \right)^{-\frac{1}{2}} - 1 \right] + \frac{1}{8} \sum_{l=0}^\infty \frac{1}{(l + \frac{1}{2})^2} \right. \\ & \times \left. \left[\frac{1}{\left(1 + \left(\frac{mr}{l + \frac{1}{2}} \right)^2 \right)^{3/2}} - \frac{6}{\left(1 + \left(\frac{mr}{l + \frac{1}{2}} \right)^2 \right)^{5/2}} + \frac{5}{\left(1 + \left(\frac{mr}{l + \frac{1}{2}} \right)^2 \right)^{7/2}} \right] \right\} \\ & + \frac{1}{4} \int_0^\infty dr r^3 V^2(r) \sum_{l=0}^\infty \frac{1}{(l + \frac{1}{2})^2} \left(1 + \left(\frac{mr}{l + \frac{1}{2}} \right)^2 \right)^{-3/2}. \end{aligned} \tag{2.36}$$

In this case, *all* terms in (2.36) cancel against corresponding terms in (2.29), after summing over l with degeneracy factor 2ν in $d = 3$. The only remaining uncanceled term in (2.29) is the first term, which is linear in V , and is the subtraction shown in (1.9). This shows that in dimension $d = 3$ we did not actually need to expand $\ln f_l^{\text{asym}}(ik)$ to $O(V^2)$ in the first place; the $O(V)$ term would have been sufficient.

The $d = 2$ case is slightly different, as we need to separate the $l = 0$ term from the sum. Here $\nu = l$, and the degeneracy factor is 1 for $l = 0$, and 2 for $l \geq 1$. Thus, instead of (2.28) we have

$$\zeta'_f(0)|_{d=2} = -\ln f_0(im) - \sum_{l=1}^\infty 2[\ln f_l(im) - \ln f_l^{\text{asym}}(im)]. \tag{2.37}$$

And in two dimensions (2.17) is

$$\zeta_{as}(s)|_{d=2} = \frac{\sin(\pi s)}{\pi} \sum_{l=1}^\infty 2 \int_m^\infty dk [k^2 - m^2]^{-s} \frac{\partial}{\partial k} \ln f_l^{\text{asym}}(ik). \tag{2.38}$$

Then the analogous computation in $d = 2$ leads to

$$\begin{aligned} \zeta'_{as}(0)|_{d=2} = & - \int_0^\infty dr r V(r) \left[\ln \left(\frac{r}{2} \right) + \gamma \right] - \int_0^\infty dr r V(r) \left\{ \sum_{l=1}^\infty \frac{1}{l} \left[\left(1 + \left(\frac{mr}{l} \right)^2 \right)^{-1/2} - 1 \right] \right. \\ & + \frac{1}{8} \sum_{l=1}^\infty \frac{1}{l^3} \left[\left(1 + \left(\frac{mr}{l} \right)^2 \right)^{-3/2} - 6 \left(1 + \left(\frac{mr}{l} \right)^2 \right)^{-5/2} + 5 \left(1 + \left(\frac{mr}{l} \right)^2 \right)^{-7/2} \right] \left. \right\} \\ & + \frac{1}{4} \int_0^\infty dr r^3 V^2(r) \sum_{l=1}^\infty \frac{1}{l^3} \left(1 + \left(\frac{mr}{l} \right)^2 \right)^{-3/2}. \end{aligned} \tag{2.39}$$

As in the $d = 4$ case, all terms involving l summation cancel exactly against identical terms in (2.29), after summing those over l , with the $d = 2$ degeneracy factors. As in $d = 3$, the only remaining uncanceled term in (2.29) is the first term, which is linear in V , and is the subtraction shown in (1.8). This shows that also in dimension $d = 2$, we did not need to expand $\ln f_l^{\text{asym}}(ik)$ to $O(V^2)$ in the first place; the $O(V)$ term would have been sufficient.

3. Comparison with Feynman diagram approach

In this section we show that our zeta function computation is equivalent to the Feynman diagrammatic expansion for the logarithm of the determinant [2, 18], although the zeta

function approach provides a much simpler form of the final expression. Consider regulating the determinant with dimensional regularization. The perturbative expansion in powers of the potential V is [2]

$$\begin{aligned} \ln \left(\frac{\det[\mathcal{M} + m^2]}{\det[\mathcal{M}^{\text{free}} + m^2]} \right) &\equiv \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} A^{(k)} \\ &= \text{circle with dot} - \frac{1}{2} \text{two circles with dots} + \frac{1}{3} \text{three circles with dots} + \dots, \end{aligned} \tag{3.1}$$

where the dots denote insertions of the potential V . Alternatively, we can expand the dimensionally regulated determinant in partial waves as

$$\begin{aligned} \ln \left(\frac{\det[\mathcal{M} + m^2]}{\det[\mathcal{M}^{\text{free}} + m^2]} \right) &= \sum_{l=0}^{\infty} \text{deg}(l; d) \ln \left(\frac{\det[\mathcal{M}_{(l)} + m^2]}{\det[\mathcal{M}_{(l)}^{\text{free}} + m^2]} \right) \\ &= \sum_{l=0}^{\infty} \text{deg}(l; d) \ln f_l(im). \end{aligned} \tag{3.2}$$

In the Feynman diagrammatic approach [18], the first two order terms, $A^{(1)}$ and $A^{(2)}$, are separated out:

$$\begin{aligned} \ln \left(\frac{\det[\mathcal{M} + m^2]}{\det[\mathcal{M}^{\text{free}} + m^2]} \right) &= \sum_{l=0}^{\infty} \text{deg}(l; d) \{ \ln f_l(im) - [\ln f_l(im)]_{O(V)} - [\ln f_l(im)]_{O(V^2)} \} \\ &\quad + A^{(1)} - \frac{1}{2} A^{(2)}. \end{aligned} \tag{3.3}$$

With these subtractions, the l sum is now finite, and the divergence lies in the dimensionally regulated Feynman diagrams $A^{(1)}$ and $A^{(2)}$. We now show how to relate the expression (3.3) to the zeta function approach.

Using dimensional regularization, the first-order Feynman diagram is

$$A^{(1)} = \int d^d x V(x) \lim_{x \rightarrow y} G(x, y). \tag{3.4}$$

The Helmholtz Green's function in d dimensions is

$$G(x, y) = \frac{m^{d-2}}{(2\pi)^{d/2}} \frac{K_{d/2-1}(m|x-y|)}{(m|x-y|)^{d/2-1}}. \tag{3.5}$$

We now use the Gegenbauer expansion [25]

$$\frac{K_\nu(|x-y|)}{|x-y|^\nu} = 2^\nu \Gamma(\nu) \sum_{l=0}^{\infty} (l+\nu) \frac{K_{l+\nu}(r)}{r^\nu} \frac{I_{l+\nu}(r')}{(r')^\nu} C_l^\nu(\cos \theta), \tag{3.6}$$

where $|x-y| = \sqrt{r^2 + (r')^2 - 2r r' \cos \theta}$. As $x \rightarrow y$, noting that $C_l^{d/2-1}(1) = \binom{l+d-3}{l}$, we find

$$\begin{aligned} A^{(1)} &= \int_0^\infty dr r V(r) \sum_{l=0}^{\infty} \frac{(2l+d-2)}{(d-2)} \binom{l+d-3}{l} K_{l+d/2-1}(mr) I_{l+d/2-1}(mr) \\ &= \sum_{l=0}^{\infty} \text{deg}(l; d) [\ln f_l(im)]_{O(V)}, \end{aligned} \tag{3.7}$$

in agreement with the $O(V)$ term in the iterative expansion for $\ln f_l(im)$ in (2.25).

Similarly, the second-order Feynman diagram is

$$A^{(2)} = \int d^d x \int d^d y V(x)G(x, y)V(y)G(y, x). \tag{3.8}$$

Using the Gegenbauer expansion (3.6), together with the identity [19]

$$\int_0^\pi d\theta (\sin \theta)^{2\nu} C_l^\nu(\cos \theta) C_l^\nu(\cos \theta) = \delta_{ll'} \frac{\pi 2^{1-2\nu} \Gamma(2\nu + l)}{l!(l + \nu)\Gamma^2(\nu)}, \tag{3.9}$$

we find

$$\begin{aligned} A^{(2)} &= 2 \sum_{l=0}^\infty \text{deg}(l; d) \int_0^\infty dr r V(r) K_{l+d/2-1}^2(mr) \int_0^r dr' r' V(r') I_{l+d/2-1}^2(mr') \\ &= -2 \sum_{l=0}^\infty \text{deg}(l; d) [\ln f_l(im)]_{O(V^2)}. \end{aligned} \tag{3.10}$$

Thus, $-\frac{1}{2}A^{(2)}$ agrees with the $O(V^2)$ term in the iterative expansion for $\ln f_l(im)$ in (2.25), when summed over l with the appropriate degeneracy factor. Therefore, the Feynman diagrammatic expression (3.3) is indeed equivalent to the zeta function expression (2.13), with dimensional regularization.

Now compare also the finite parts. In the Feynman diagrammatic approach [18], the finite renormalized logarithm of the determinant ratio is defined by the subtractions in (3.3), together with the finite renormalized form of the first two Feynman diagrams, $A^{(1)}$ and $A^{(2)}$. For definiteness we consider the $d = 4$ case, in order to compare with previous work [16, 18]. Then the renormalized logarithm of the determinant ratio is [18]

$$\begin{aligned} \ln \left(\frac{\det[\mathcal{M} + m^2]}{\det[\mathcal{M}^{\text{free}} + m^2]} \right) &= \sum_{l=0}^\infty (l + 1)^2 \{ \ln f_l(im) - [\ln f_l(im)]_{O(V)} - [\ln f_l(im)]_{O(V^2)} \} \\ &+ A_{\text{fin}}^{(1)} - \frac{1}{2} A_{\text{fin}}^{(2)} = \sum_{l=0}^\infty (l + 1)^2 \left\{ \ln \left(\frac{\psi_{(l)}(\infty)}{\psi_{(l)}^{\text{free}}(\infty)} \right) - [\ln f_l(im)]_{O(V)} \right. \\ &\left. - [\ln f_l(im)]_{O(V^2)} \right\} + A_{\text{fin}}^{(1)} - \frac{1}{2} A_{\text{fin}}^{(2)}, \end{aligned} \tag{3.11}$$

where we have used (2.14) to identify $\ln f_l(im)$ with $\psi_{(l)}(\infty)/\psi_{(l)}^{\text{free}}(\infty)$. In [18], the subtraction terms are defined as

$$[\ln f_l(im)]_{O(V)} \equiv h_l^{(1)}(\infty), \quad [\ln f_l(im)]_{O(V^2)} \equiv h_l^{(2)}(\infty) - \frac{1}{2}(h_l^{(1)}(\infty))^2, \tag{3.12}$$

where $h_l^{(k)}(r)$ is the solution to the differential equation

$$\begin{aligned} \left[\frac{d^2}{dr^2} + \left(\frac{2mI'_{l+1}(mr)}{I_{l+1}(mr)} + \frac{1}{r} \right) \frac{d}{dr} \right] h_l^{(k)}(r) &= V(r)h_l^{(k-1)}(r) \\ h_l^{(k)}(0) = 0, \quad h_l^{(k)'}(0) = 0, \quad h_l^{(0)} &\equiv 1. \end{aligned} \tag{3.13}$$

Comparing with (2.25), we can alternatively express these subtracted terms as

$$\begin{aligned} [\ln f_l(im)]_{O(V)} &= \int_0^\infty dr r V(r) K_\nu(mr) I_\nu(kr) \\ [\ln f_l(im)]_{O(V^2)} &= - \int_0^\infty dr r V(r) K_\nu^2(mr) \int_0^r dr' r' V(r') I_\nu^2(mr'). \end{aligned} \tag{3.14}$$

Finally, the finite contributions in (3.11) from the first- and second-order Feynman diagrams in the \overline{MS} scheme are [18] (note the small typo in equation (4.32) of [18])

$$\begin{aligned} A_{\text{fin}}^{(1)} &= -\frac{m^2}{8} \int_0^\infty dr r^3 V(r) \\ A_{\text{fin}}^{(2)} &= \frac{1}{128\pi^4} \int_0^\infty dq q^3 |\tilde{V}(q)|^2 \left[2 - \frac{\sqrt{4m^2 + q^2}}{q} \ln \left(\frac{\sqrt{4m^2 + q^2} + q}{\sqrt{4m^2 + q^2} - q} \right) \right], \end{aligned} \quad (3.15)$$

where $\tilde{V}(q)$ is the four-dimensional Fourier transform of the radial potential $V(r)$.

With the terms subtracted in (3.11) (evaluated using either (3.13) or (3.14)), the l sum is convergent. So this expression yields a finite answer for the logarithm of the determinant. On the other hand, the actual subtraction terms and counterterms in (3.11) are different from those in (1.10), even though the final net answer for the finite renormalized determinant is numerically the same. The difference between the two approaches is that in (3.11) one subtracts the *full* $O(V)$ and $O(V^2)$ dependence of $\ln f_l(im)$, given in (2.25), and then compensates this subtraction with the Feynman diagram counter-terms whose finite part in the \overline{MS} scheme are given in (3.15). On the other hand, in the zeta function computation, we subtract just the *asymptotic form* of these first two Feynman diagrams, as in (2.27), as is required to analytically continue the zeta function to $s = 0$. Subsequently, in the zeta function approach we note that even this is an over-subtraction, as part of this asymptotic behaviour cancels against $\zeta'_{as}(0)$, leaving the final expression (1.10). The regularized form of these zeta function subtractions is different from the regularized form of the Feynman diagrammatic subtractions, but the associated counter-terms are also different, in such a way that the net result for the determinant is identical. This follows analytically from (3.7) and (3.10), and can easily be confirmed numerically. This also serves as an explanation of similar effects noted in one-loop static energy computations [26, 27]. However, we note that the zeta function expression (1.10) has a significantly simpler form, with the subtractions only requiring simple integrals involving $V(r)$, while the subtractions in (3.11) require the more complicated integrals (3.14) (or, equivalently, solving the differential equations in (3.13)), and also require the Fourier transform of the potential in (3.15).

4. Conclusions

To conclude, we have derived simple new expressions, (1.8)–(1.10), for the determinant of a radially separable partial differential operator of the form $-\Delta + m^2 + V(r)$, generalizing the Gel'fand–Yaglom result (1.3) to higher dimensions. This greatly increases the class of differential operators for which the determinant can be computed simply and efficiently. Our derivation uses the zeta function definition of the determinant, but the same expressions can be found using the radial WKB approach of [16, 17]. Furthermore, we have shown how these expressions relate to the Feynman diagrammatic definition of the determinant based on dimensional regularization [18]. These superficially different expressions are in fact equal, although the zeta function expression is considerably simpler.

A number of generalizations could be made. First, in certain quantum field theory applications the determinant may have zero modes, and correspondingly one is actually interested in computing the determinant with these zero modes removed. Our method provides a simple way to compute such determinants. For example, in the false vacuum decay problem [28–31] arising in a self-interacting scalar field theory in d -dimensional spacetime, the prefactor for the semiclassical decay rate involves the functional determinant of the fluctuation operator for quantum fluctuations about a radial classical bounce solution $\Phi_{cl}(r)$. This

fluctuation operator is a radially separable operator of the form considered in this paper, but there is a d -fold degenerate zero mode in the $l = 1$ sector, associated with translational invariance of the classical bounce solution. It is straightforward to generalize the analysis of [16] for the $d = 4$ case to other dimensions, to find that the net prefactor contribution from these $l = 1$ zero modes (including the collective coordinate factor [3, 32]) has a simple expression solely in terms of the asymptotic behaviour of the bounce solution:

$$\left(\frac{S[\Phi_{cl}]}{2\pi}\right)^{d/2} \left(\frac{\det'[\mathcal{M}_{(l=1)} + m^2]}{\det[\mathcal{M}_{(l=1)}^{\text{free}} + m^2]}\right)^{-1/2} = [(2\pi)^{d/2-1} \Phi_\infty |\Phi_{cl}''(0)|]^{d/2}. \quad (4.1)$$

Here the constant Φ_∞ is defined by the normalization of the asymptotic large r behaviour of the bounce: $\Phi_{cl}(r) \sim \Phi_\infty r^{1-d/2} K_{d/2-1}(r)$. Another important generalization is to include *directly* the matrix structure that arises from Dirac-like differential operators and from non-Abelian gauge degrees of freedom. The Feynman diagrammatic approach is well developed for such separable problems [33]; for example it has been applied to the fluctuations about the electroweak sphaleron [34, 35], and to compute the metastability of the electroweak vacuum [36]. More recently, the angular momentum cut-off method has been used to compute the full mass dependence of the fermion determinant in a four-dimensional Yang–Mills instanton background [17], to compute the fermion determinant in a background instanton in the two-dimensional chiral Higgs model [37], and to address the fluctuation problem for false vacuum decay in curved space [38]. A unified zeta function analysis should be possible, as there is a straightforward generalization of the Gel'fand–Yaglom result (1.3) to systems of ordinary differential operators [14].

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